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On the Thermal Stresses in a Finite Circular Cylinder*

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SUMMARY

A solution is presented for the determination of thermal stresses in a finite cylinder heated axisymmetrically over the curved surface. The solution is obtained by constructing the thermoelastic displacement potential and the biharmonic Love function to satisfy all the boundary conditions. It considers the steady state stresses as well as the transient stresses.

1. Introduction

Thermal stresses are of vital concern in many design problems of engineering. In the literature, writings on thermal stresses in a cylinder have been abundant. Many authors [1, 2, 3, 6, 7, 12, 14] have given solutions for thermal stresses in an infinite cylinder with particular thermal boundary conditions. When the solution obtained for an infinite cylinder is used to determine thermal stresses in a finite cylinder, the stresses at the end surfaces usually are only self-equilibrating instead of vanishing. Aiming to eliminate these non-vanishing end stresses, quite a few authors [4, 5, 9, 10] have considered the end problem of a finite cylinder with self-equilibrating end stresses in isothermal elasticity.

A direct and more general solution is expounded here for the determination of thermal stresses in a finite cylinder. The solution has been formulated for a hollow cylinder and takes into account both the transient and the steady state stresses. The solution consists of two parts: the thermoelastic displacement potential, which is a particular solution; and the biharmonic Love function, which is the homogeneous solution. They combine to satisfy all the boundary conditions.

2. Basic Equations

The cylindrical coordinate system (r, θ, z) will be used as the reference frame. The inner radius of the cylinder is a, the outer radius b, and the height $2h$. The end surfaces are at $z = \pm h$. Because of axial symmetry, all the equations will be independent of the coordinate θ.

The temperature distribution $T(r, z, t)$ is governed by the equation of heat conduction in a solid body as the following:

$$\nabla^2 T = \frac{1}{\chi} T_{,t} \tag{1}$$

where χ is the coefficient of diffusivity and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

The displacement field (u, w) is governed by the Navier displacement equations

$$\nabla^2 u - \frac{u}{r^2} + \frac{\left(u_{,r} + \dfrac{u}{r} + w_{,z}\right)_{,r}}{1-2\nu} = \frac{2(1+\nu)}{1-2\nu}\alpha T_{,r} \tag{2a}$$

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$$\nabla^2 w + \frac{\left(u_{,r} + \frac{u}{r} + w_{,z}\right)_{,z}}{1-2\nu} = \frac{2(1+\nu)}{1-2\nu}\,\alpha T_{,z}\,, \tag{2b}$$

where ν is the Poisson ratio and α the coefficient of thermal expansion.

A particular solution of (2) is represented by the thermoelastic displacement potential Φ satisfying ([11]):

$$\nabla^2 \Phi = kT \tag{3}$$

where $k = \alpha(1+\nu)/(1-\nu)$. The homogeneous solution of (2), i.e., the solution of (2) when the right-hand sides are zero, will be represented by the Love function Ψ satisfying

$$\nabla^2 \nabla^2 \Psi = 0\,. \tag{4}$$

The functions Φ and Ψ shall be so constructed that they combine to satisfy all the boundary conditions and thus completely define the displacement field. Since the stresses σ_{ij} will be specified on the boundary, they need to be expressed in terms of Φ and Ψ and can be derived, respectively, from

$$\begin{aligned}
\bar{\sigma}_{rr} &= -2G\left(\frac{1}{r}\Phi_{,r} + \Phi_{,zz}\right)\\
\bar{\sigma}_{zz} &= -2G\left(\Phi_{,rr} + \frac{1}{r}\Phi_{,r}\right)\\
\bar{\sigma}_{\theta\theta} &= -2G\left(\Phi_{,rr} + \Phi_{,zz}\right)\\
\bar{\sigma}_{rz} &= -2G\Phi_{,rz}
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\bar{\bar{\sigma}}_{rr} &= 2G\left[\nu\nabla^2\Psi - \Psi_{,rr}\right]_{,z}\\
\bar{\bar{\sigma}}_{zz} &= 2G\left[(2-\nu)\nabla^2\Psi - \Psi_{,zz}\right]_{,z}\\
\bar{\bar{\sigma}}_{\theta\theta} &= 2G\left[\nu\nabla^2\Psi - \frac{1}{r}\Psi_{,r}\right]_{,z}\\
\bar{\bar{\sigma}}_{rz} &= 2G\left[(1-\nu)\nabla^2\Psi - \Psi_{,zz}\right]_{,r}\,,
\end{aligned} \tag{6}$$

with G the modulus of rigidity.

3. Solution of the Heat Conduction Equation

The thermal boundary conditions are (with $\alpha_n = n\pi/h$ and $H(t)$ the unit step function):

$$\begin{aligned}
T &= H(t)\left[\sum_{n=1}^{\infty} T_n^i \cos\alpha_n z + T_0^i\right] \quad &&\text{at} \quad r=a\\
T &= H(t)\left[\sum_{n=1}^{\infty} T_n^0 \cos\alpha_n z + T_0^0\right] \quad &&\text{at} \quad r=b\\
\frac{\partial T}{\partial z} &= 0\,. &&\text{at} \quad z=\pm h\,.
\end{aligned} \tag{7}$$

The initial temperature of the cylinder is assumed to zero, i.e.,

$$T = 0 \quad \text{at} \quad t = 0\,. \tag{8}$$

The solution of (1) satisfying (7) and (8) can be shown to be

$$T = T^c(r) + T'(r, z) + T''(r, z, t)\,. \tag{9}$$

Here

$$T^c(r) = T_0^0 + (T_0^i - T_0^0)\frac{\log(b/r)}{\log(b/a)} \tag{10a}$$

and

$$\begin{aligned} T'(r, z) = \sum_{n=1}^{\infty} [\{T_n^i K_0(\alpha_n b) - T_n^0 K_0(\alpha_n a)\} I_0(\alpha_n r) \\ + \{T_n^0 I_0(\alpha_n a) - T_n^i I_0(\alpha_n b)\} K_0(\alpha_n r)] \frac{\cos \alpha_n z}{\Delta_n} \end{aligned} \tag{10b}$$

is the steady-state temperature distribution;

$$\begin{aligned} T''(r, z, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\pi \beta_m^2 \exp[-(\beta_m^2 + \alpha_n^2)\chi t]\{T_n^i J_0(\beta_m b) - T_n^0 J_0(\beta_m a)\}}{(\beta_m^2 + \alpha_n^2)\{J_0^2(\beta_m a) - J_0^2(\beta_m b)\}} \times \\ \times \{J_0(\beta_m r) Y_0(\beta_m a) - Y_0(\beta_m r) J_0(\beta_m a)\} J_0(\beta_m b) \cos \alpha_n z \end{aligned} \tag{11}$$

is the transient temperature variation; the β_m are the roots of

$$J_0(bx) Y_0(ax) - Y_0(bx) J_0(ax) = 0 \tag{12}$$

and

$$\Delta_n = I_0(\alpha_n a) K_0(\alpha_n b) - I_0(\alpha_n b) K_0(\alpha_n a). \tag{13}$$

4. Stresses Due to the Steady-State Temperature Distribution

The stress boundary conditions are

$$\sigma_{rz} = 0 \qquad \sigma_{zz} = 0 \qquad \text{at } z = \pm h \tag{14}$$

$$\sigma_{rz} = 0 \qquad \sigma_{rr} = 0 \qquad \text{at } r = a \tag{15}$$

$$\sigma_{rz} = 0 \qquad \sigma_{rr} = 0 \qquad \text{at } r = b \tag{16}$$

When T in (3) is $T'(r, z)$ of (10b), the function Φ may have the form

$$\Phi' = \frac{k}{2} \sum_{n=1}^{\infty} \left[\frac{L_n I_1(\alpha_n r)}{I_0(\alpha_n b)} + \frac{M_n K_1(\alpha_n r)}{K_0(\alpha_n a)} \right] \frac{r \cos \alpha_n z}{\alpha_n} \tag{17}$$

where

$$\begin{aligned} L_n &= \frac{I_0(\alpha_n b)}{\Delta_n} \{T_n^i K_0(\alpha_n b) - T_n^0 K_0(\alpha_n a)\} \\ M_n &= -\frac{K_0(\alpha_n a)}{\Delta_n} \{T_n^0 I_0(\alpha_n a) - T_n^i I_0(\alpha_n b)\}. \end{aligned} \tag{18}$$

The corresponding stresses are, from (5),

$$\begin{aligned} \bar{\sigma}_{rr} = -Gk \sum_{n=1}^{\infty} \left[\frac{L_n}{I_0(\alpha_n b)} \{I_0(\alpha_n r) - \alpha_n r I_1(\alpha_n r)\} \right. \\ \left. - \frac{M_n}{K_0(\alpha_n a)} \{K_0(\alpha_n r) + \alpha_n r K_1(\alpha_n r)\} \right] \cos \alpha_n z \end{aligned}$$

$$\bar{\sigma}_{zz} = -Gk \sum_{n=1}^{\infty} \left[\frac{L_n I_0(\alpha_n r)}{I_0(\alpha_n b)} - \frac{M_n K_0(\alpha_n r)}{K_0(\alpha_n a)} \right] \cos \alpha_n z$$

$$\bar{\sigma}_{\theta\theta} = -Gk \sum_{n=1}^{\infty} \left[\frac{L_n}{I_0(\alpha_n b)} \{2I_0(\alpha_n r) + \alpha_n r I_1(\alpha_n r)\} - \frac{M_n}{K_0(\alpha_n a)} \{2K_0(\alpha_n r) - \alpha_n r K_1(\alpha_n r)\} \right] \cos \alpha_n z$$

$$\bar{\sigma}_{rz} = -Gk \sum_{n=1}^{\infty} \left[\frac{L_n I_0(\alpha_n r)}{I_0(\alpha_n b)} - \frac{M_n K_0(\alpha_n r)}{K_0(\alpha_n a)} \right] \alpha_n r \sin \alpha_n z \,. \tag{19}$$

The temperature variation $T^c(r)$ is due to the difference in the mean temperatures of the outer and the inner surfaces. Stresses corresponding to $T^c(r)$ are from [13].

$$\hat{\sigma}_{rr} = \frac{Gk(T_0^i - T_0^0)}{\log(b/a)} \left[-\log\left(\frac{b}{r}\right) - \frac{a^2}{(b^2 - a^2)} \left(1 - \frac{b^2}{r^2}\right) \log\left(\frac{b}{a}\right) \right]$$

$$\hat{\sigma}_{\theta\theta} = \frac{Gk(T_0^i - T_0^0)}{\log(b/a)} \left[1 - \log\left(\frac{b}{r}\right) - \frac{a^2}{(b^2 - a^2)} \left(1 + \frac{b^2}{r^2}\right) \log\left(\frac{b}{a}\right) \right] \tag{20}$$

$$\hat{\sigma}_{zz} = \frac{Gk(T_0^i - T_0^0)}{\log(b/a)} \left[1 - 2\log\frac{b}{r} - \frac{2a^2}{(b^2 - a^2)} \log\left(\frac{b}{a}\right) \right]$$

It may be observed that the stresses given by (19) and (20) satisfy only the first condition of (14). Then it is necessary to construct a Ψ function so that the remaining boundary conditions can be fulfilled. The Ψ function may have the following form:

$$\Psi' = \sum_{n=1}^{\infty} \left[A_n \Psi_{1n} + B_n \Psi_{2n} + C_n \Psi_{3n} + Gk \Psi_{0n} \right] . \tag{21}$$

The functions Ψ_{mn} in (21) are given by

$$\Psi_{1n} = \left[z \cosh \lambda_n z - \left(\frac{2\nu}{\lambda_n} + h \coth \lambda_n h \right) \sinh \lambda_n z \right] \frac{F_0(\lambda_n r)}{\lambda_n^2 \cosh \lambda_n h}$$

$$\Psi_{2n} = \left[\left\{ r I_1(\alpha_n r) - \left(\frac{2(1-\nu)}{\alpha_n} + b \frac{I_0(\alpha_n b)}{I_1(\alpha_n b)} \right) I_0(\alpha_n r) \right\} \Big/ I_1(\alpha_n b) + \frac{\Pi_n}{K_1(\alpha_n a)} \left\{ r K_1(\alpha_n r) + \left(\frac{2(1-\nu)}{\alpha_n} - b \frac{K_0(\alpha_n b)}{K_1(\alpha_n b)} \right) K_0(\alpha_n r) \right\} \right] \frac{\sin \alpha_n z}{\alpha_n^2} \tag{22}$$

$$\Psi_{3n} = \left[\frac{\Gamma_m}{I_1(\alpha_n b)} \left\{ r I_1(\alpha_n r) - \left(\frac{2(1-\nu)}{\alpha_n} + a \frac{I_0(\alpha_n a)}{I_1(\alpha_n a)} \right) I_0(\alpha_n r) \right\} + \left\{ r K_1(\alpha_n r) + \left(\frac{2(1-\nu)}{\alpha_n} - a \frac{K_0(\alpha_n a)}{K_1(\alpha_n a)} \right) K_0(\alpha_n r) \right\} \Big/ K_1(\alpha_n a) \right] \frac{\sin \alpha_n z}{\alpha_n^2}$$

and

$$\Psi_{0n} = \left[\frac{L_n}{I_0(\alpha_n b)} \left\{ r I_1(\alpha_n r) - \frac{2(1-\nu)}{\alpha_n} I_0(\alpha_n r) \right\} + \frac{M_n}{K_0(\alpha_n a)} \left\{ r K_1(\alpha_n r) + \frac{2(1-\nu)}{\alpha_n} K_0(\alpha_n r) \right\} \right] \frac{\sin \alpha_n z}{\alpha_n^2} , \tag{23}$$

where

$$F_0(\lambda_n r) = J_0(\lambda_n r) + \mu_n Y_0(\lambda_n r)$$

$$\Pi_n = \frac{aI_0(\alpha_n a)/I_1(\alpha_n b) - bI_0(\alpha_n b)I_1(\alpha_n a)/I_1^2(\alpha_n b)}{aK_0(\alpha_n a)/K_1(\alpha_n a) - bK_0(\alpha_n b)/K_1(\alpha_n b)}$$

$$\Gamma_n = \frac{bK_0(\alpha_n b)/K_1(\alpha_n a) - aK_0(\alpha_n a)K_1(\alpha_n b)/K_1^2(\alpha_n a)}{bI_0(\alpha_n b)/I_1(\alpha_n b) - aI_0(\alpha_n a)/I_1(\alpha_n a)}. \tag{24}$$

The λ_n are the roots of

$$J_1(\lambda a)\,Y_1(\lambda b) - J_1(\lambda b)\,Y_1(\lambda a) = 0\,, \tag{25}$$

and the μ_n are given by $-J_1(\lambda_n a)/Y_1(\lambda_n a)$ or $-J_1(\lambda_n b)/Y_1(\lambda_n b)$.

Substitution from (21) into (6) will yield a set of stresses corresponding to Ψ'. This set of stresses will be denoted by $\bar{\bar{\sigma}}_{ij}$. Then the total steady-state stresses are obtained as

$$\sigma'_{ij} = \hat{\sigma}_{ij} + \bar{\sigma}_{ij} + \bar{\bar{\sigma}}_{ij}\,. \tag{26}$$

It is now noted that σ'_{rz} will vanish on all boundaries because of the way with which Ψ' has been constructed. Thus only three conditions, i.e., the second equations of (14)–(16), remain to be satisfied. Applying the boundary conditions $\sigma'_{rr}(a, z) = 0$ and taking the finite Fourier transformation of the resulting equation will eventually lead to

$$\sum_{m=1}^{\infty} P_{nm} F_0(\lambda_n a) A_m + Q_n^0 B_n + R_n^0 C_n = \frac{2Gk(1-\nu)}{\alpha_n a}\left[\frac{L_n}{J_0(\alpha_n b)} I_1(\alpha_n a) + \frac{M_n}{K_0(\alpha_n a)} K_1(\alpha_n a)\right], \qquad n = 1, 2, 3, \ldots \tag{27}$$

where

$$P_{nm} = \frac{(-1)^n 4\lambda_m \alpha_n^2}{h(\lambda_m^2 + \alpha_n^2)^2} \tanh \lambda_m h$$

$$Q_n^0 = \left\{ I_0(\alpha_n a)\left(1 + \alpha_n b\frac{I_0(\alpha_n b)}{I_1(\alpha_n b)}\right) - \alpha_n a I_1(\alpha_n a) - \left(\frac{2(1-\nu)}{\alpha_n a} + \frac{bI_0(\alpha_n b)}{aI_1(\alpha_n b)}\right) I_1(\alpha_n a)\right\} \Big/ I_1(\alpha_n b)$$
$$- \Pi_n \left\{\left(1 - \alpha_n b\frac{K_0(\alpha_n b)}{K_1(\alpha_n b)}\right)\frac{K_0(\alpha_n a)}{K_1(\alpha_n a)} + \alpha_n a + \frac{2(1-\nu)}{\alpha_n a} - \frac{b}{a}\frac{K_0(\alpha_n b)}{K_1(\alpha_n b)}\right\}$$

$$R_n^0 = \frac{\Gamma_n}{I_1(\alpha_n b)}\left\{\alpha_n a\frac{I_0^2(\alpha_n a)}{I_1(\alpha_n a)} - \left(\alpha_n a + \frac{2(1-\nu)}{\alpha_n a}\right) I_1(\alpha_n a)\right\} - \alpha_n a + \alpha_n a\frac{K_0^2(\alpha_n a)}{K_1^2(\alpha_n a)} - \frac{2(1-\nu)}{\alpha_n a}. \tag{28}$$

Similarly, the boundary condition $\sigma'_{rr}(b, z) = 0$ will yield

$$\sum_{m=1}^{\infty} P_{nm} F_0(\lambda_m b) A_m + Q_n B_n + R_n C_n = \frac{2(1-\nu)Gk}{\alpha_n b}\left[\frac{L_n}{I_0(\alpha_n b)} I_1(\alpha_n b) + \frac{M_n}{K_0(\alpha_n a)} K_1(\alpha_n b)\right], \qquad n = 1, 2, 3, \ldots \tag{29}$$

where

$$Q_n = \alpha_n b\frac{I_0^2(\alpha_n b)}{I_1^2(\alpha_n b)} - \alpha_n b - \frac{2(1-\nu)}{\alpha_n b} - \frac{\Pi_n}{K_1(\alpha_n a)}\left\{\left(\alpha_n b + \frac{2(1-\nu)}{\alpha_n b}\right) K_1(\alpha_n b) - \alpha_n b\frac{K_0^2(\alpha_n b)}{K_1(\alpha_n b)}\right\}$$

$$R_n = \Gamma_n\left\{\left(1 + \alpha_n a\frac{I_0(\alpha_n a)}{I_1(\alpha_n a)}\right)\frac{I_0(\alpha_n b)}{I_1(\alpha_n b)} - \alpha_n b - \frac{2(1-\nu)}{\alpha_n b} - \frac{a}{b}\frac{I_0(\alpha_n a)}{I_1(\alpha_n a)}\right\}$$
$$- \left\{\alpha_n b K_1(\alpha_n b) + \left(1 - \alpha_n a\frac{K_0(\alpha_n a)}{K_1(\alpha_n a)}\right) K_0(\alpha_n b) + \left(\frac{2(1-\nu)}{\alpha_n b} - \frac{a}{b}\frac{K_0(\alpha_n a)}{K_1(\alpha_n a)}\right) K_1(\alpha_n b)\right\} \Big/ K_1(\alpha_n a)\,. \tag{30}$$

Finally, setting $\sigma'_{zz}(r, \pm h)$ of (26) to zero and taking the finite Bessel–Fourier transformation of the resulting equation will lead to

$$\left[1+\frac{2\lambda_m h}{\sinh 2\lambda_m h}\right]A_m+\sum_{n=1}^{\infty}\left[Q_{mn}B_n+R_{mn}C_n\right](-1)^n=g_m \qquad m=1,2,3,\ldots \tag{31}$$

where

$$\begin{aligned}Q_{mn}&=\left\{\alpha_n I^1_{mn}+\left(\frac{2}{\alpha_n}-b\frac{I_0(\alpha_n b)}{I_1(\alpha_n b)}\right)\alpha_n I^0_{mn}\right\}\Big/ I_1(\alpha_n b)\\&\quad+\Pi_n\left\{\alpha_n K^1_{mn}-\left(\frac{2}{\alpha_n}+b\frac{K_0(\alpha_n b)}{K_1(\alpha_n b)}\right)\alpha_n K^0_{mn}\right\}\Big/ K_1(\alpha_n a)\\R_{mn}&=\Gamma_n\left\{\alpha_n I^1_{mn}+\left(2-\alpha_n a\frac{I_0(\alpha_n a)}{I_1(\alpha_n a)}\right)I^0_{mn}\right\}\Big/ I_1(\alpha_n b)\\&\quad+\left\{\alpha_n K^1_{mn}-\left(2+\alpha_n a\frac{K_0(\alpha_n a)}{K_1(\alpha_n a)}\right)K^0_{mn}\right\}\Big/ K_1(\alpha_n a)\end{aligned} \tag{32}$$

and

$$g_m=-\frac{4Gk(T^i_0-T^0_0)\{F_0(\lambda_m b)-F_0(\lambda_m a)\}}{\lambda_m^2\log(b/a)\{b^2F_0^2(\lambda_m b)-a^2F_0^2(\lambda_m a)\}} \tag{33}$$

The I^0_{mn}, I^1_{mn}, K^0_{mn}, and K^1_{mn} in (32) are given in Appendix 1.

Equations (27), (29), and (31) are the necessary relations to be satisfied by the unknown constants A_m, B_n, and C_n such that all the stress boundary conditions may be completely satisfied. Once A_m, B_n, and C_n are obtained from (27), (29), and (31), the steady-state stresses are totally defined and given by (26).

5. Stresses Due to the Transient Part of the Temperature Variation

$T''(r,z,t)$ of (11) is first expanded into a Bessel–Fourier series as follows:

$$T''(r,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}T_{mn}(t)F_0(\lambda_m r)\cos\alpha_n z \tag{34}$$

where

$$\begin{aligned}T_{mn}(t)&=\frac{2}{h\{b^2F_0^2(\lambda_m b)-a^2F_0^2(\lambda_m a)\}}\int_a^b\int_{-h}^{h}rT''(r,z,t)F_0(\lambda_m r)\cos\alpha_n z\,dr\,dz\\&=\frac{4}{b^2F_0^2(\lambda_m b)-a^2F_0^2(\lambda_m a)}\sum_{s=1}^{\infty}\frac{\beta_s^2\exp[-(\beta_s^2+\alpha_n^2)\chi t]\{T^i_nJ_0(\beta_s b)-T^0_nJ_0(\beta_s a)\}}{(\beta_s^2+\alpha_n^2)(\beta_s^2-\lambda_m^2)\{J_0^2(\beta_s a)-J_0^2(\beta_s b)\}}\\&\qquad\times\left\{J_0(\beta_s a)F_0(\lambda_m b)-\frac{b}{a}F_0(\lambda_m a)J_0(\beta_s b)\right\}.\end{aligned} \tag{35}$$

Then the thermoelastic displacement potential is assumed to be

$$\Phi''(r,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(t)F_0(\lambda_m r)\cos\alpha_n z\,. \tag{36}$$

Substitution of (34) and (36) into (3) results in

$$\phi_{mn}(t)=-\frac{T_{mn}(t)k}{\lambda_m^2+\alpha_n^2}\,. \tag{37}$$

Stresses due to $\Phi''(r,z,t)$ are computed from (5).

$$\bar{\sigma}''_{rr}=-2Gk\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{T_{mn}(t)}{\lambda_m^2+\alpha_n^2}\left[\lambda_m\frac{F_1(\lambda_m r)}{r}+\alpha_n^2F_0(\lambda_m r)\right]\cos\alpha_n z$$

$$\bar{\sigma}''_{zz} = -2Gk \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda_m^2 T_{mn}(t)}{\lambda_m^2 + \alpha_n^2} F_0(\lambda_m r) \cos \alpha_n z$$

$$\bar{\sigma}''_{\theta\theta} = -2Gk \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{T_{mn}(t)}{\lambda_m^2 + \alpha_n^2} \left[-\lambda_m \frac{F_1(\lambda_m r)}{r} + (\lambda_m^2 + \alpha_n^2) F_0(\lambda_m r) \right] \cos \alpha_n z$$

$$\sigma''_{rz} = -2Gk \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda_m \alpha_n T_{mn}(t)}{\lambda_m^2 + \alpha_n^2} F_1(\lambda_m r) \sin \alpha_n z \, . \tag{38}$$

The Love function may be assumed to be

$$\Psi''(r, z, t) = \sum_{n=1}^{\infty} [A''_n(t) \Psi_{1n}(r, z) + B''_n(t) \Psi_{2n}(r, z) + C''_n(t) \Psi_{3n}(r, z)] \tag{39}$$

where Ψ_{1n}, Ψ_{2n}, and Ψ_{3n} are given by (22), and A''_n, and B''_n, and C''_n are functions of time. Analytic expressions of these functions in general are not obtainable, but they may be determined numerically. The stresses corresponding to $\Psi''(r, z, t)$ can be computed from (6) and will be denoted by $\bar{\bar{\sigma}}''_{ij}(r, z, t)$. Then the total transient stresses, i.e., stresses corresponding to $T''(r, z, t)$, will be

$$\sigma''_{ij}(r, z, t) = \bar{\sigma}''_{ij}(r, z, t) + \bar{\bar{\sigma}}''_{ij}(r, z, t) \, . \tag{40}$$

The σ''_{ij} decrease as time increases, and are negligible when time becomes fairly large.

Since the total stresses in the cylinder are

$$\sigma_{ij}(r, z, t) = \sigma'_{ij}(r, z) + \sigma''_{ij}(r, z, t) \tag{41}$$

and both σ_{ij} and σ'_{ij} vanish on the boundary, it follows that the σ''_{ij} must also satisfy (14)–(16). It is noted that σ''_{rz} vanishes on the boundary because of the way Φ'' and Ψ'' have been constructed. Then only the second equations of (14)–(16) remain to be fulfilled and they are used to determine A'', B'', and C''. Equations similar to (27), (29), and (31) will be obtained by satisfying $\sigma''_{rr}(a, z, t) = 0$, $\sigma''_{rr}(b, z, t) = 0$, and $\sigma''_{zz}(r, \pm h, t) = 0$, respectively. They are as follows:

$$\sum_{m=1}^{\infty} P_{nm} F_0(\lambda_m a) A''_m(t) + Q_n^0 B''_n(t) + R_n^0 C''_n(t) = 2 \sum_{m=1}^{\infty} \frac{T_{mn}(t) \alpha_n^2}{\lambda_m^2 + \alpha_n^2} F_0(\lambda_m a) \qquad n = 1, 2, 3, \ldots \tag{42}$$

$$\sum_{m=1}^{\infty} P_{nm} F_0(\lambda_m b) A''_m(t) + Q_n B''_n(t) + R_n C''_n(t) = 2 \sum_{m=1}^{\infty} \frac{T_{mn}(t) \alpha_n^2}{\lambda_m^2 + \alpha_n^2} F_0(\lambda_m b) \qquad n = 1, 2, 3, \ldots \tag{43}$$

$$\left[1 + \frac{2\lambda_m h}{\sinh 2\lambda_m h} \right] A''_m(t) + \sum_{n=1}^{\infty} [Q_{mn} B''_n(t) + R_{mn} C''_n(t)] (-1)^n = 2 \sum_{n=1}^{\infty} \frac{\lambda_m^2 T_{mn}(t) (-1)^n}{\lambda_m^2 + \alpha_n^2} \qquad m = 1, 2, 3, \ldots \tag{44}$$

Time appears in equations (42)–(44) as a parameter. These equations can be solved for any particular time. Hence, functions A'', B'', and C'' can be evaluated numerically. Once they are determined, the transient stresses σ''_{ij} corresponding to the temperature variation T'' will be completely defined.

6. Numerical Examples

To illustrate the procedure presented in the previous sections, two numerical examples are given. All the computations have been performed with the aid of an IBM 360 computer.

1. *A Hollow Cylinder Heated on the Inner Surface*

Only stationary stresses are considered. The temperature of the inner surface of the cylinder is kept constant (T_0). All the Fourier coefficients in (7) vanish except T_0^i; $T_n^i = T_n^0 = 0$, $n = 1, 2, 3, \ldots$, $T_0^0 = 0$, and $T_0^i = T_0$. The steady-state temperature becomes simply

$$T^c(r) = T_0 \log(b/r) / \log(b/a) . \tag{45}$$

Now (27), (29), and (31) can be solved simultaneously to yield values of A_n, B_n, and C_n and thus Ψ' is determined. Then stresses are obtained according to (26).

For a long cylinder, stresses at sections not too close to the end have been given by Timoshenko [13]. If the cylinder has a small height-to-diameter ratio, the stresses obtained for a cylinder cease to be accurate at any cross section because of the end effect. Thus the variations of stresses, obtained with the procedure presented in this paper, at section $z=0$ (the middle cross section) are shown in Fig. 1 and Fig. 2 for a cylinder with $a/b=0.3$, $\nu=0.3$ and various h/b.

The axial stresses σ'_{zz}, at the section $z=0$ are plotted in Fig. 1. It can be seen that the axial stress decreases as the ratio h/b decreases. This fact shows the tendency that the state of stress is approaching that of plane stress where the axial stress is very small if not vanishing. On the other hand, the axial stress approaches to that given by the long cylinder solution when h/b becomes greater than 1.5.

The circumferential stresses $\sigma'_{\theta\theta}$, also the section $z=0$, are plotted in Fig. 2. When $h/b>2$, the $\sigma_{\theta\theta}$ given by the present method is indistinguishable from that given by Timoshenko solution. However, unlike the σ'_{zz}, the $\sigma'_{\theta\theta}$ doesn't approach the $\sigma_{\theta\theta}$ of the state of plane stress monotonically as h/b decreases. The compression on the inner surface and the tension in the outer surface seem to attain their maximum values respectively when the ratio h/b is somewhere between 0.5 and 1.5.

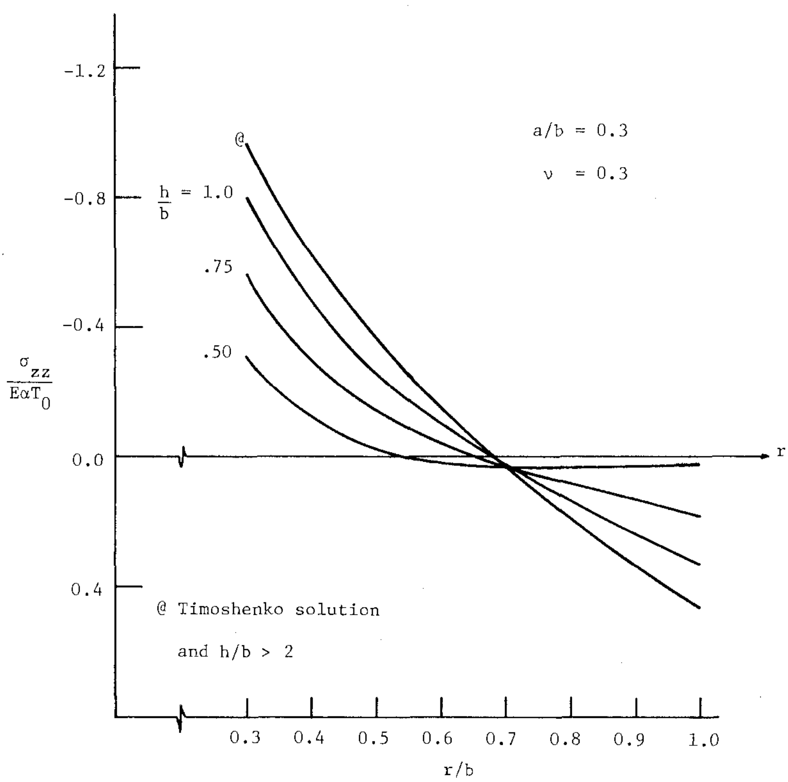

Fig. 1. Distribution of axial stresses at section $z=0$ of a hollow cylinder with constant temperature T_0 on the inner surface and zero temperature on the outer surface.

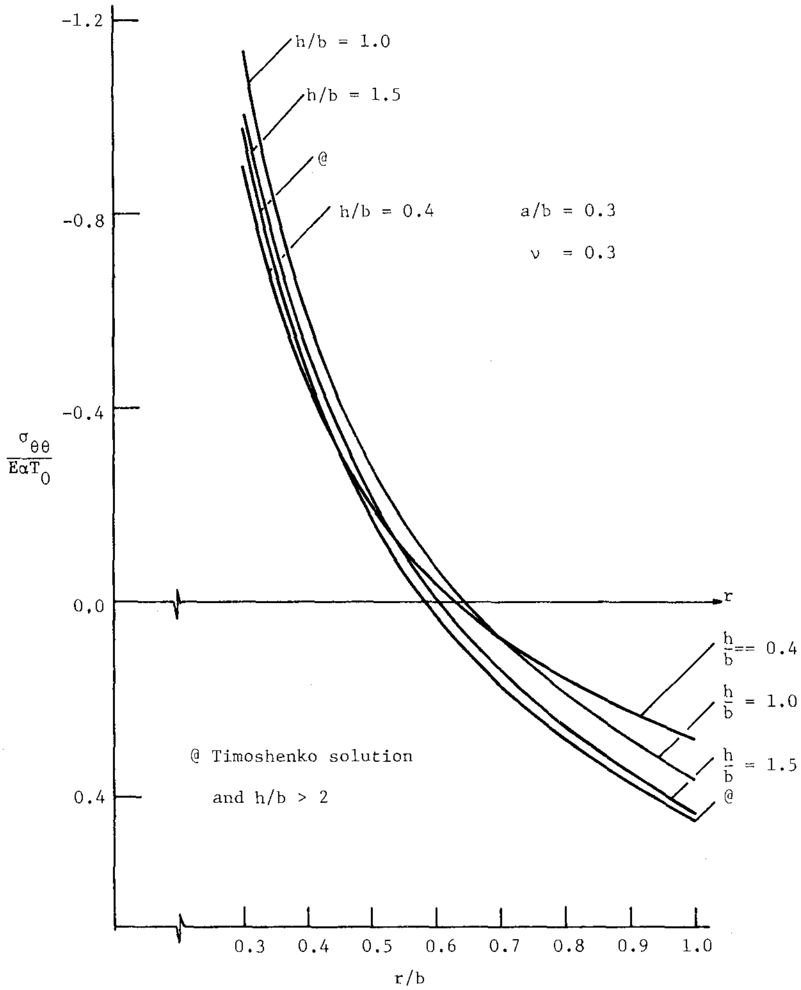

Fig. 2. Distribution of circumferential stresses at section $z=0$ of a hollow cylinder with constant temperature T_0 on the inner surface and zero temperature on the outer surface.

These variations of the circumference stresses and the axial stresses are apparently in agreement with the often-made assumption that the end effects are negligible at a distance of one diameter from the end.

2. *Cylinder Heated Over a Band on the Middle Portion of the Outer Surface*

Only the steady-state stresses will be given. The thermal boundary conditions for this case are

$$T(a, z) = 0 \tag{46}$$

$$\begin{aligned} T(b, z) &= T_0\,, \qquad 0 < |z| < c \\ &= 0\,, \qquad |z| > c \end{aligned} \tag{47}$$

where T_0 is a constant and $c/h = 0.125$. The Fourier coefficients in (7) will now have the values

$$T_n^i = 0 \qquad n = 0, 1, 2, \ldots$$

$$T_n^0 = \frac{1}{h}\int_{-h}^{+h} T(b, z)\cos\alpha_n z\, dz = \frac{2T_0}{n\pi}\sin(n\pi c/h) \qquad n = 1, 2, 3, \ldots \tag{48}$$

$$T_0^0 = 0.125\ T_0\,.$$

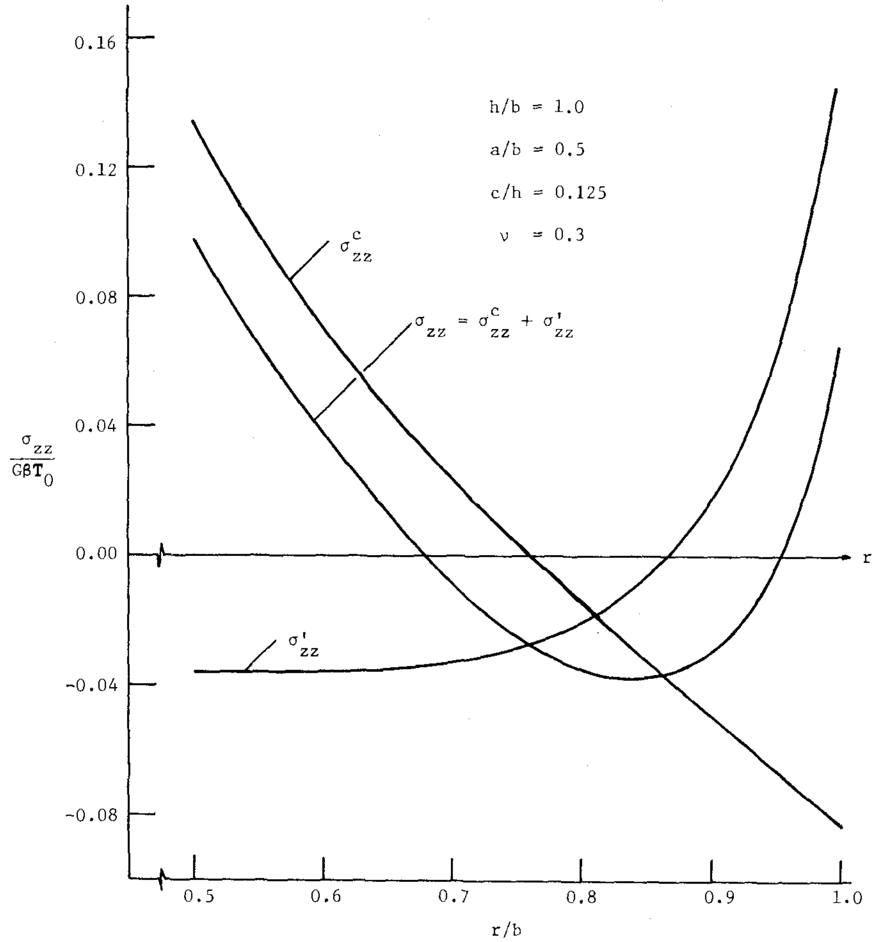

Fig. 3. Distribution of axial stresses at section $z=0$ of a hollow cylinder when the middle portion of the outer surface is heated to maintain a constant temperature (T_0) band of width $2c$.

The results of computation are presented in Fig. 3–8. The total steady-state stresses are separated into two parts: the σ'_{ij} corresponding to $T'(r, z)$ and the σ^c_{ij} corresponding to T^c, which is independent of z; i.e., $\sigma_{ij}=\sigma'_{ij}+\sigma^c_{ij}$. Since the section $z=0$ is most critical, stress variations along it are shown in Fig. 3–5. It can be seen that the σ'_{ij} contributes more significantly to the σ_{ij} than the σ^c_{ij} does and that the circumferential stress $\sigma_{\theta\theta}$ is much larger compared to either the axial stress or the radial stress. Hence variations of the circumferential stress $\sigma'_{\theta\theta}$ along different longitudinal sections are shown in Fig. 6–7; the $\sigma^c_{\theta\theta}$ remains constant along any longitudinal section. The sudden jump of $\sigma'_{\theta\theta}$ at $z/h=0.125$ on the outer surface is due to the discontinuity of the surface temperature. A similar jump has been observed in the solution of stresses in an infinite solid cylinder heated over a band [14].

7. Concluding Remarks

The procedure presented in this paper may be used to determine directly thermal stresses in a finite cylinder with temperature prescribed on the curved surface and no surface tractions. All the boundary conditions as well as the governing differential equations can be satisfied. Although only the solution with symmetry in z has been given, the anti-symmetric part can be similarly formulated [15] with no difficulty.

It is noted that the steady-state solution given here reduces to that for a solid cylinder given by Iyengar [8] when the inner radius approaches zero [15]. The Love function may also be

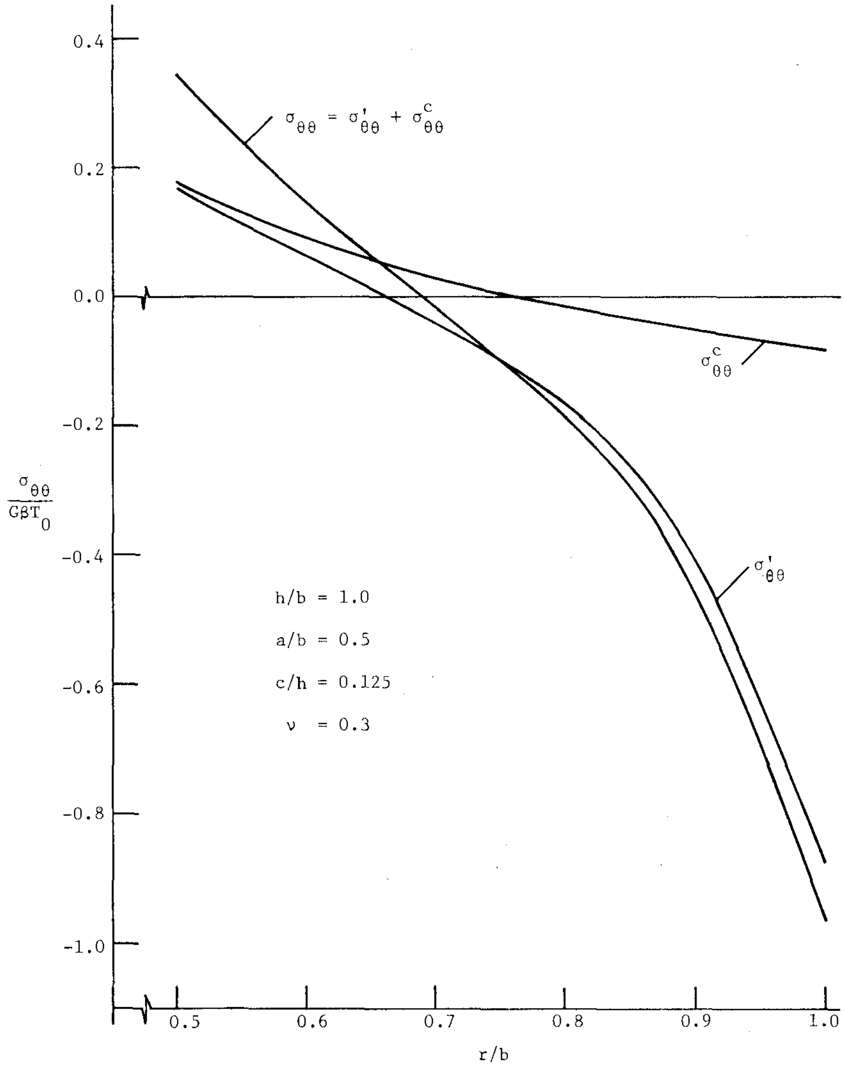

Fig. 4. Distribution of circumferential stresses at section $z=0$ of a hollow cylinder when the middle portion of the outer surface is heated to maintain a constant temperature (T_0) band of width $2c$.

used in the solution of the end problem of a finite hollow cylinder in isothermal elasticity.

Appendix

The function $F_0(\lambda_l r) = J_0(\lambda_l r) + \mu_l Y_0(\lambda_l r)$ is the solution of the differential equation

$$\frac{d^2 y}{dr^2} + \frac{1}{r}\frac{dy}{dr} + \lambda_l^2 y = 0 \tag{A.1}$$

with the boundary conditions

$$\frac{dy}{dr} = 0\,, \quad r = a \text{ and } r = b\,. \tag{A.2}$$

Thus $F_0(\lambda_l r)$ form a complete orthogonal set i.e.,

$$\int_a^b rF_0(\lambda_l r)\, F_0(\lambda_m r)\, dr = 0\,, \qquad l = m\,,$$

and

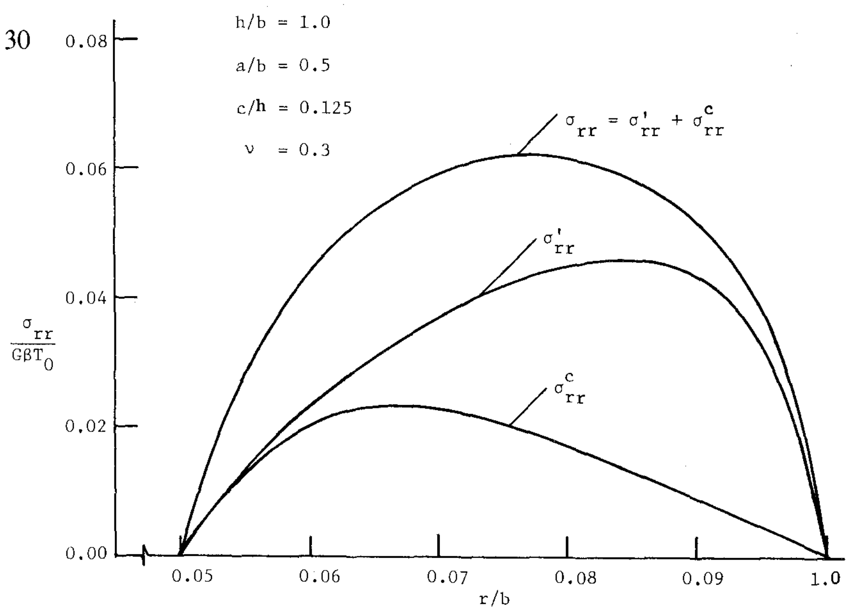

Fig. 5. Distribution of radial stresses at section $z=0$ of a hollow cylinder when the middle portion of the outer surface is heated to maintain a constant temperature (T_0) band of width $2c$.

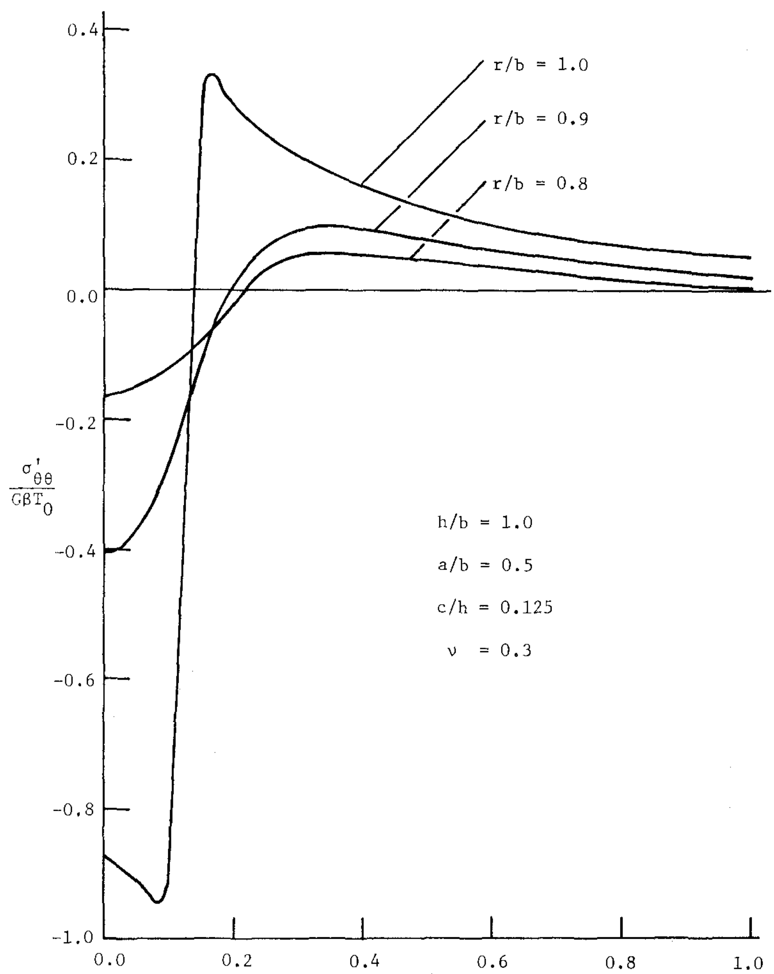

Fig. 6. Distribution of circumferential stress $\sigma'_{\theta\theta}$ at various vertical sections of a hollow cylinder when the middle portion of the outer surface is heated to maintain a constant temperature (T_0) band of width $2c$.

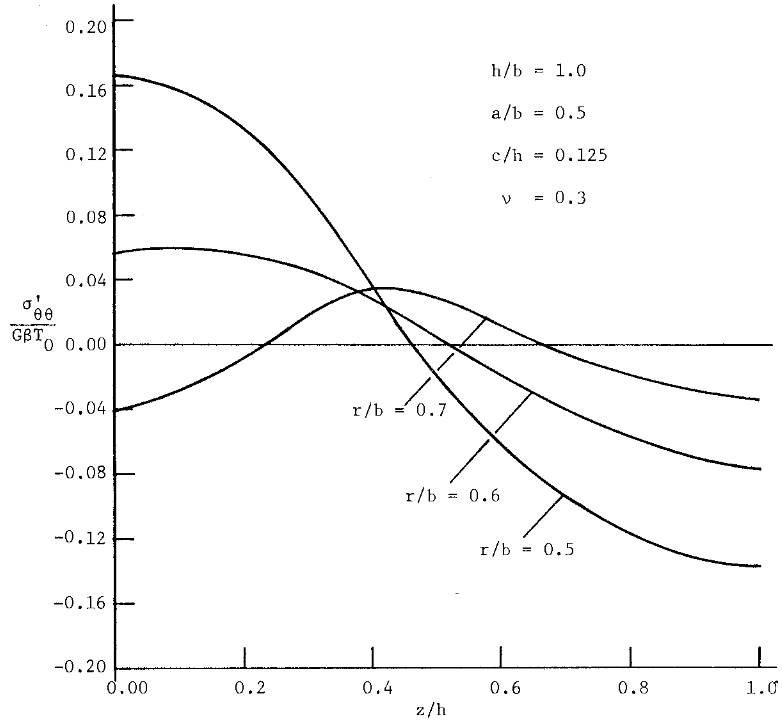

Fig. 7. Distribution of circumferential stress $\sigma'_{\theta\theta}$ at various vertical sections of a hollow cylinder when the middle portion of the outer surface is heated to maintain a constant temperature (T_0) band of width $2c$.

$$\Omega = \int_a^b rF_0^2(\lambda_l r)dr = \tfrac{1}{2}\{bF_0(\lambda_l b)\}^2 - \tfrac{1}{2}\{aF_0(\lambda_l a)\}^2 . \tag{A.3}$$

Hence the functions $I_0(\alpha_n r)$, $rI_1(\alpha_n r)$, $K_0(\alpha_n r)$, and $rK_1(\alpha_n r)$ may be represented as

$$\begin{aligned} I_0(\alpha_n r) &= \sum_{l=1}^{\infty} I_{nl}^0 F_0(\lambda_l r) , \\ rI_1(\alpha_n r) &= \sum_{l=1}^{\infty} I_{nl}^1 F_0(\lambda_l r) , \\ K_0(\alpha_n r) &= \sum_{l=1}^{\infty} K_{nl}^0 F_0(\lambda_l r) , \\ rK_1(\alpha_n r) &= \sum_{l=1}^{\infty} K_{nl}^1 F_0(\lambda_l r) , \end{aligned} \tag{A.4}$$

where

$$I_{nl}^0 = \frac{1}{\Omega}\int_a^b rI_0(\alpha_n r)F_0(\alpha_l r)dr = \frac{n}{\Omega(a_n^2+\lambda_l^2)}\{bF_0(\lambda_l b)I_1(\alpha_n b) - aF_0(\lambda_l a)I_1(\alpha_n a)\} ,$$

$$\begin{aligned} I_{nl}^1 &= \frac{1}{\Omega}\int_a^b r^2 I_1(\alpha_n r)F_0(\lambda_l r)dr \\ &= \frac{\alpha_n}{\Omega(\alpha_n^2+\lambda_l^2)}\{b^2 I_0(\alpha_n b)F_0(\lambda_l b) - a^2 I_0(\alpha_n a)F_0(\lambda_l a)\} \\ &\quad - \frac{2\alpha_n^2}{\Omega(\alpha_n^2+\lambda_l^2)}\{bF_0(\lambda_l b)I_1(\alpha_n b) - aF_0(\lambda_l a)I_1(\alpha_n a)\} , \end{aligned}$$

$$K_{nl}^0 = \frac{1}{\Omega}\int_a^b rK_0(\alpha_n r)F_0(\lambda_l r)dr$$

$$= \frac{\alpha_n}{\Omega(\lambda_l^2+\alpha_n^2)}\left[-bF_0(\lambda_l b)K_1(\alpha_n b)+aF_0(\lambda_l a)K_1(\alpha_n a)\right], \tag{A.5}$$

$$K_{nl}^1 = \frac{1}{\Omega}\int_a^b r^2 K_1(\alpha_n r)F_0(\lambda_l r)dr$$

$$= \frac{\alpha_n}{\Omega(\lambda_l^2+\alpha_n^2)}\left\{-b^2 K_0(\alpha_n b)F_0(\lambda_l b)+a^2 K_0(\alpha_n a)F_0(\lambda_l a)\right\}$$

$$+ \frac{2\alpha_n}{\Omega(\alpha_n^2+\lambda_l^2)^2}\left\{-bF_0(\lambda_l b)K_1(\alpha_n b)+aF_0(\lambda_l a)K_1(\alpha_n a)\right\}.$$

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